

Universality of the Berezinskii-Kosterlitz-Thouless type of phase transition in a planar gas of dipoles

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We investigate the nature of the phase transition occurring in a planar XY-model spin system with dipole-dipole interactions. It is demonstrated that a Berezinskii-Kosterlitz-Thouless (BKT) type of phase transition always occurs at a finite temperature, separating the ordered (ferro) and the disordered (para) phases. The low-temperature phase corresponds to an ordered state with thermal fluctuations, composed of a “gas” of bound vortex-antivortex pairs, which would, when considered isolated, be characterized by a constant vortex-antivortex attraction force which is due to the dipolar interaction term in the Hamiltonian. Using a topological charge model, we show that small bound vortex pairs are easily polarized, and screen the vortex-antivortex interaction in sufficiently large pairs. Screening changes the linear attraction potential of vortices to a logarithmic one, and leads to the familiar pair dissociation mechanism of the BKT type. The topological charge model is confirmed by numerical calculations, in which we demonstrate that the transition temperature slightly increases when compared with the standard BKT result for short-range interactions.

The standard paradigm of phase transitions in a planar system of electrically neutral particles representing an effective XY-model spin dictates that long-range order does not exist at any finite temperature [1, 2]. On the other hand, below a critical temperature T_c , spin-spin correlations in two spatial dimensions decay in a power-law fashion. As a consequence, short-range cooperative phenomena, such as superfluidity, can exist at these low temperatures. Below the critical point, thermal excitations predominantly occur in the form of vortex-antivortex pairs (VAPs). Due to the attraction between vortex and antivortex, the VAPs remain bound at $T < T_c$, and the gas thus remains superfluid, while the dissociation of VAPs at $T > T_c$ leads to an exponential decay of correlations. This standard scenario, corresponding to the celebrated BKT phase transition [3–5], however conventionally only applies when the particles interact by short-range (contact) interactions.

Spin ordering phenomena in planar systems with long-range forces mediated by the dipole-dipole interaction (DDI) between particles carrying static dipole moments μ , have been extensively studied in the past, cf., e.g., [6–26]. It was found that the DDI leads to a stabilization of the long-range ferromagnetic order against thermal fluctuations, and the ground state of the spin system may thus be spontaneously polarized [6–12], or acquire various structures [14–26]. So far most efforts aimed at understanding the phase transition (PT) were a combination of renormalization group (RG) arguments with phenomenological approaches [7–18], or Monte Carlo and Molecular Dynamics simulations [19–24]. However, it is fair to say that a complete picture of the nature of the PT in the dipolar XY model so far has been elusive. There is, in

particular, no agreement on both the physical mechanism of the PT in the dipolar XY model and its critical temperature.

For contact interactions, vortex and antivortex in a VAP interact by a logarithmic interaction potential, which in superfluids is due to kinetic energy of the flow, and the attraction force decays with the inverse distance. This logarithmic interaction is the primary requirement for the occurrence of the BKT transition. On the other hand, in the presence of a dipolar interaction term in the Hamiltonian, a spatially constant attraction force K_0 between vortex and antivortex occurs [8, 11, 12] (also see below), which bears an obvious potential importance for the PT, which has been overlooked in most previous investigations of the dipolar XY model. We note that the *confinement* in isolated VAPs by the string tension is one of the rare instances outside the realm of Quantum Chromodynamics (QCD), in which linear interaction potentials between a particle and its antiparticle, in QCD between quark and antiquark, occur.

The existence of vortex confinement led to the statement [11, 12] that a novel PT distinct from BKT takes place in the two-dimensional (2D) dipolar gas. In the following, we critically examine this conclusion. We demonstrate that the sole effect of the DDI is that a VAP dissociation transition of the BKT type occurs at slightly higher temperatures. To this end, we use an analytical model, applicable sufficiently close to the transition point, backed up by numerical simulations for the full range of temperatures. We show that the apparently inconsistent pictures of confinement of isolated VAPs and occurrence of a PT driven by the familiar BKT mechanism of pair dissociation can be made fully consistent

with each other if one correctly accounts for shielding of the bare VAP tension at finite temperatures. The shielding effect of the linear attraction potential within a large pair is mediated by the large number of small VAPs in which it is immersed. The vortex-antivortex interaction only remains linear at small distances R between the vortices in a pair, $R < r_0$, where the length parameter r_0 is defined in Eq. (7) below; it however becomes logarithmic at sufficiently large VAP size, $R > r_0$.

The polarization (effective spin) states of the dipolar XY model, which correspond to a VAP gas, are described by the continuous two-component vector-field $\mathbf{s}(\mathbf{r}) = s \cos \theta \mathbf{e}_x + s \sin \theta \mathbf{e}_y$, representing the hydrodynamic, coarse-grained average of the spin, where θ is the azimuthal angle. In dimensionless form, the hydrodynamic free energy functional is $G_S[\mathbf{s}(\mathbf{r})] = G_0 + G_{dd}$. The short-ranged term G_0 corresponds to a perturbation of the uniformly polarized ground state ($a, b = x, y$) [27]:

$$G_0 = \frac{1}{2} \int_{\Gamma} df \left[\sum_{a,b} \nabla_a s_b \nabla_a s_b + (s^2 - 1)^2 \right], \quad (1)$$

where df is the surface element, while the dipolar interaction energy functional is given by [6, 7, 11, 12]

$$G_{dd} = \Lambda \int_{\Gamma} df df' \frac{\rho_P(\mathbf{r}) \rho_P(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (2)$$

Here $\rho_P(\mathbf{r}) = -\nabla \cdot \mathbf{s}(\mathbf{r})$ is the density of polarization charges, and $\Lambda \propto \mu^2$ a dimensionless interaction constant (representing the DDI coupling constant). The appropriate units of energy and length depend on the specific system considered; as its value does not affect our argument, we have also put the “contact interaction” coefficient of the $(s^2 - 1)^2$ term equal to unity. We assume that the gas of dipoles is stable against collapse due to the long-range attractive part of the DDI (head to tail instability) at any value of Λ because of a sufficiently large short-range repulsive core of the true interaction potential. Examples for possible applications of the present model span a broad range, including the commonly studied thin magnetic [14] and ferroelectric [26] films, quantum gases [28–31], but also, in a biological context, water molecules on hydrated proteins [32].

Focusing our attention on the regime $T \approx T_c$, we have, typically, that the VAP size $R \gg 1$. For these VAPs, large compared to a microscopic (e.g. coherence length) scale, using that $\nabla \cdot \mathbf{s} \sim 1/R$ and hence $G_0 \sim \log R$, $G_{dd} \sim R$. Therefore, the energy of a VAP with $R \gg 1$ increases linear with their size, and vortex and antivortex attract each other with a constant force [8, 11, 12], in sharp distinction to contact interactions, where the force decreases linearly as the VAP size increases. We then restate the total free energy of the system in the form

($i, j = 1 \dots 2N$, where N is the number of VAPs) [11, 12]

$$G_S \approx -\frac{1}{2} \sum_{i,j} q_i q_j u(r_{ij}) = \frac{1}{2} \sum_i q_i \Phi(\mathbf{r}_i). \quad (3)$$

The function $u(r) \approx 2\pi \log(1 + \alpha r) + K_0 r$, and $q_i = \pm 1$ are the topological charges of the vortices [4, 5] located at $\mathbf{r}_j = (x_j, y_j)$, $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. The quantity $K_0 \equiv K_0(\Lambda)$ is the “VAP tension” coefficient, depending on the coupling strength Λ . The logarithmic contribution in $u(r)$ arises from G_0 in Eq. (1) and describes the interaction of vortices in the limit of $\Lambda \rightarrow 0$ [3–5]. The parameter α describes interaction between vortices at $r \rightarrow 0$ and when $\Lambda = 0$; it is found from numerical calculations of the VAP energy (see below). At $r \rightarrow 0$ vortices of a given pair annihilate, corresponding to the requirement $u(0) = 0$. The “potential” $\Phi(\mathbf{r})$ in (3) is introduced in analogy with electrostatics:

$$\Phi(\mathbf{r}) = \sum_j q_j F(\mathbf{r} - \mathbf{r}_j) = \int F(\mathbf{R}) \rho(\mathbf{r}') d^2 \mathbf{r}'. \quad (4)$$

Here, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $F(\mathbf{R}) = -2\pi \ln(1 + \alpha R) - K_0 R$, and the vortex topological charge density $\rho(\mathbf{r}) = \sum_j q_j \delta^{(2)}(\mathbf{r} - \mathbf{r}_j)$. The topological charge inside a 2D contour C is defined as usual to be $Q = (2\pi)^{-1} \int_C d\theta$.

Eq. (3) is the central assumption leading to the phenomenological description of our model system. We have set up the corresponding Langevin Dynamics simulations and ran extensive checks to establish the (nontrivial) validity of Eq. (3) (see supplementary material). Numerical calculations also let us establish the functional dependence of $K_0(\Lambda)$. We found that at $\Lambda \gtrsim \Lambda_{CR} \simeq 0.4$, corresponding to $K_0(\Lambda_{CR}) \approx 3.1$, a physical instability of the single VAP configuration arises: A new small VAP is spontaneously created in the center of a large VAP. With further increase of Λ , the “parent” vortices are immersed into a cloud of small polarized VAPs that decrease the effective topological charge of the parent vortices. According to (4), the topological charge decrease is equivalent to K_0 increasing with Λ , finally saturating at $K_0(\Lambda) \approx K_0(\Lambda_{CR}) \approx 3.1$ (see the inset of Fig. 2). The spontaneous creation of pairs follows also from the expression (3): At $\Lambda > \Lambda_{CR}$ the energy of the large ($R \gg 1$) pair decreases with the emergence of a new small pair in the center. This occurs at $\Lambda = \Lambda_{CR}$, if we choose $\alpha = K_0(\Lambda_{CR})/2\pi \approx K_0(\infty)/2\pi$. Typical numerically calculated (see supplementary material) vector field $\mathbf{s}(\mathbf{r})$ configurations and total energy distributions at $T = 0$ and close to T_c are depicted in Fig. 1, illustrating the screening effect for the linear attraction by the small VAPs close to the PT.

Once the additive expression for the energy (3) is verified by the numerics, we can use it to elucidate the nature of the PT and to derive an analytical expression for the transition temperature T_c . By its

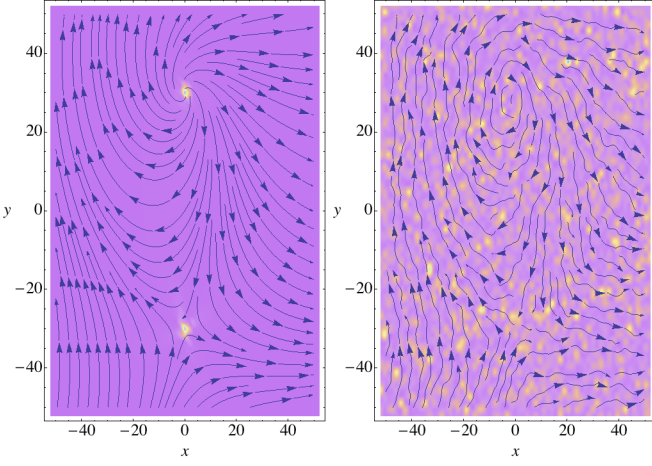


Figure 1: *Left:* Polarization field $\mathbf{s}(\mathbf{r})$ and total energy distribution at $T = 0$ for a VAP located at $(0, \pm 30)$. Brighter regions characterize larger energy density, where vortices or VAPs are located. The angle between the axis of the large VAP and the asymptotic uniform polarization is chosen to be 45° . *Right:* Configuration at temperatures close to T_c , where a large number of thermally excited small VAPs strongly alters the power law of attraction in the given VAP at $(0, \pm 30)$.

definition, the potential $\Phi(\mathbf{r})$ satisfies a Poisson type equation, $\hat{L}_\mathbf{r}\Phi(\mathbf{r}) = \rho(\mathbf{r})$. Here the linear operator $\hat{L}_\mathbf{r}$ is defined such that $\hat{L}_\mathbf{r}F(\mathbf{r} - \mathbf{r}') = \delta^{(2)}(\mathbf{r} - \mathbf{r}')$, which in Fourier representation reads $L_\mathbf{k} = 1/F_\mathbf{k} = \{2\pi K_0 k^{-3} + 4\pi^2 \alpha / [k^2(k + \alpha)]\}^{-1}$. Let us follow the electrostatic analogy further. Since the energy of a charge (vortex) q placed in the external potential Φ is $U = q\Phi$, the force, acting on the charge (vortex) is $\mathbf{F} = q\mathbf{E}$, where the quasi-electric field vector $\mathbf{E} = -\nabla\Phi$. The energy of the pair in this field is $U = q_+\Phi(\mathbf{r}_+) + q_-\Phi(\mathbf{r}_-) \approx -\mathbf{d} \cdot \mathbf{E}$. Here the topological dipole moment of a pair is introduced according to $\mathbf{d} = \sum_j q_j \mathbf{r}_j = q_+ \mathbf{r}_+ + q_- \mathbf{r}_- \equiv \mathbf{r}$, where $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$. The density of the topological polarization charges is $\rho_{TP} = -\nabla \cdot \mathbf{P}$, where $\mathbf{P}(\mathbf{r}) = n_P \langle \mathbf{d} \rangle$ is polarization vector of the VAP gas, and n_P is surface density of VAPs; $\langle \dots \rangle$ denotes statistical averaging. After averaging the polarization topological charge inside a contour C , $Q_{TP} = \int df \rho_{TP}$, becomes a fractional number in general, while before averaging it must be integer.

The equation for the potential of a point charge Q placed at the origin is $\Phi_Q(\mathbf{r})$ is $\hat{L}_\mathbf{r}\Phi_Q(\mathbf{r}) = Q\delta^{(2)}(\mathbf{r}) + \rho_{TP}$. In the Weak Field Approximation (WFA), that is keeping only the linear terms in $\nabla\Phi$ of Eq. (5), we obtain a general formula for $\mathbf{P}(\mathbf{r})$:

$$\mathbf{P}(\mathbf{r}) = - \int df' \psi(|\mathbf{r} - \mathbf{r}'|) \nabla\Phi(\mathbf{r}'), \quad (5)$$

Hence $(\rho_{TP})_\mathbf{k} = -k^2\psi(\mathbf{k})$, where $\psi(\mathbf{k}) = \int df \exp(-i\mathbf{k} \cdot \mathbf{r}) \psi(\mathbf{r})$. Since $\psi(\mathbf{r})$ decays fast enough with r , then $\chi \equiv \psi(\mathbf{k} = 0) = \int df \psi(\mathbf{r})$ is a finite number, which leads to $(\rho_{TP})_\mathbf{k} \rightarrow -\chi k^2 \Phi_Q \mathbf{k}$ at $k \rightarrow 0$. Note that

for the slowly varying field $\mathbf{P}(\mathbf{r}) \approx \chi \mathbf{E}(\mathbf{r}) = -\chi \nabla\Phi(\mathbf{r})$. Therefore χ is the *susceptibility of the VAP gas*. Putting everything together, we have for the potential

$$\Phi_Q(\mathbf{r}) = Q \int \frac{d^2k}{(2\pi)^2} \frac{\exp(i\mathbf{k} \cdot \mathbf{r})}{L_\mathbf{k} + \chi k^2}. \quad (6)$$

We define the distance scale

$$r_0 = \frac{1}{2\pi K_0 \chi}, \quad (7)$$

which involves the VAP tension K_0 and the susceptibility χ of the VAP gas. At large distances, $r \gg r_0$, the dominant contribution to the integral comes from small values of k , for which the $L_\mathbf{k}$ term in the denominator is negligible. Therefore, since $(\ln r)_\mathbf{k} = -2\pi/k^2$, the potential of the “charge” at large distances is logarithmic: $\Phi_Q(\mathbf{r}) = -(Q/2\pi\chi) \ln(r/C_1)$, $r \gg r_0$, where the constant $C_1 \sim r_0$. In the opposite limit, the potential is linear: $\Phi_Q(\mathbf{r}) \approx -QK_0 r$, $r \ll r_0$. We propose a simple interpolating expression: $\Phi_Q(\mathbf{r}) \approx -(Q/2\pi\chi) \ln(1 + r/r_0)$, so that the energy of a sufficiently large pair of size R is given by

$$U(R) = -\Phi_{Q=1}(R) = \frac{1}{2\pi\chi} \ln\left(1 + \frac{R}{r_0}\right). \quad (8)$$

The distance scale r_0 in Eq. (7) represents an analogue of the Debye screening radius for 2D interactions of topological charges. Indeed, the polarization topological charge density of VAP gas next to the single topological charge Q equals: $\rho_{TP} = -\nabla \cdot \mathbf{P} \approx -\chi Q K_0 / r$. We conclude that the total charge inside a circle of radius R is given by $Q_t = Q + Q_{TP} \approx Q(1 - 2\pi K_0 \chi R)$. From this qualitative consideration we thus come to an important conclusion: the charge is essentially *shielded*, $Q_t \approx 0$, which occurs at a distance scale $R \sim r_0$. The polarization of the VAP gas inhibits the linear attraction within large VAPs, which would prevail with shielding not taken into account. Hence we conclude that the PT associated with the dissociation of pairs is qualitatively very similar to the BKT transition in a system with $\Lambda \rightarrow 0$.

The standard calculation of the transition temperature for a gas of polarizable VAPs with interaction (8) gives the following implicit equation for the transition temperature in terms of the susceptibility [3–5],

$$T_c = \frac{1}{4\pi\chi}. \quad (9)$$

At the transition temperature, $T = T_c$, VAPs begin to dissociate. This implies that at $T_c - T \ll T_c$ only a small fraction of the pairs is large and close to dissociation. For this reason it is possible to neglect the interactions between the largest VAPs and calculate the energy of a single large pair approaching its dissociation limit, which is permeated by a cloud of comparatively small

bound VAPs. Hereinafter we will subdivide VAPs into two classes: small pairs (SP) and large, close to dissociation, pairs (LP). The shielding effect arises due to the polarization cloud provided by SPs, influencing the potential energy of a LP. Let us calculate first the polarizability α_P of a single SP. The energy of the SP in an external field \mathbf{E} equals $V(\mathbf{r}) = u(r) - \mathbf{r} \cdot \mathbf{E}$. The average dipole moment of the SP and the susceptibility of the VAP gas within the framework of the WFA are given by the relations $\langle \mathbf{d} \rangle = \int d\mathbf{r} \exp(-V/T) / \int d\mathbf{r} \exp(-V/T) \approx \alpha_P \mathbf{E}$, $\chi = \alpha_P n_P$. Here, $\alpha_P = I_3 / (2TI_1)$ is the SP polarizability we are looking for, $I_n = 2\pi \int_0^\infty dr r^n e^{-\gamma r} (1 + \alpha r)^{-\beta}$, $\beta = 2\pi/T$, $\gamma = K_0/T$. The potential energy of a SP in the field of the charge Q , $\langle \mathbf{d} \rangle \nabla \Phi \approx \langle \mathbf{d} \rangle \nabla \Phi_V(\rho)$, approximately does not depend on the position of the pair, ρ , so that n_P can be considered as a constant.

Using Eq.(8), the average interaction between the vortices at finite temperature is $\langle U(R) \rangle \equiv (2\pi\chi)^{-1} \ln(1 + R_{DP}/r_0) = z_P^{-1} \int_0^\infty dR U(R) g(R) = T_c (1 - 4\tau) / [(-\tau)(1 - 2\tau)]$, where $z_P = \int_0^\infty dR g(R)$, $g(R) = R \exp[-U(R)/T]$, and the relative temperature $\tau = (T - T_c)/T_c$. Therefore, next to the PT, $|\tau| \ll 1$, the dimension of a dissociated pair is large, $R_{DP} \sim r_0 \exp(1/|\tau|)$, and its topological “electric” field is small, $\mathbf{E} = -\nabla U(R) \propto \exp(-1/|\tau|)$. A condition for the applicability of the WFA is therefore the existence of a small parameter $\exp(-1/|\tau|)$, representing the ratio of the typical topological “electric” field, $\sim |\nabla U(R)|$ to the field inside the SP, $\sim |\nabla u(R)|$. Therefore, the WFA is applicable close to the transition temperature.

The surface density of VAPs, n_P , at T_c can be calculated as follows. At $T \approx T_c$ the VAPs only start to dissociate and the fraction of LPs is small. The typical pair is small, and the interaction $\approx u(r)$ between its vortices is still unscreened. The partition function of N pairs on a surface with area S is given by $z = z_1^N / N! \approx (z_1 e / N)^N$, with $z_1 = \int d^2\rho d^2r \exp(-u/T)$. Minimization of the free energy $F_P(N) = -T \log z$ gives $N = z_1$, $n_P = N/S = I_1$. Using this result and the relation $T_c = (4\pi\chi)^{-1}$, there follows $2\pi I_3 = 1$, leading to an implicit equation for the critical temperature

$$4\pi^2 \int_0^\infty dr \frac{r^3 \exp(-K_0 r/T_c)}{(1 + \alpha r)^{2\pi/T_c}} = 1. \quad (10)$$

Next we compare the semianalytical solution for T_c from (10) with the results of the numerical calculation, cf. Fig.2. As described in detail in the supplementary material, we used Langevin Dynamics and Binder’s method to calculate T_c in a series of simulations of increasingly larger realizations of the model system. In the non-interacting case, $K_0 = 0$ (or $\Lambda = 0$), the equation (10) yields $T_c \approx 0.7$. From our numerical calculations, we found that $T_c = 0.85$. At large Λ the transition temperature tends to the constant value $T_c(\infty) \approx 1.4$. From (10)

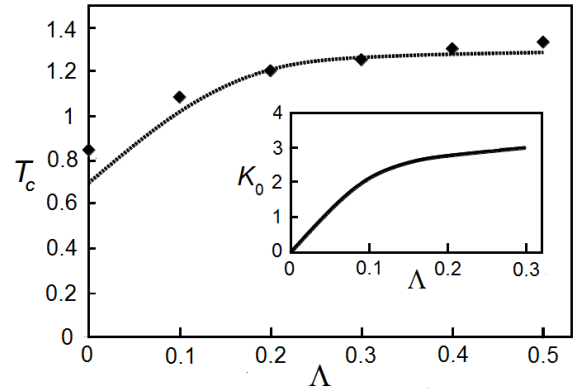


Figure 2: Transition temperature T_c versus DDI coupling strength Λ computed using Eq. (10) (dashed line), and numerical result for T_c (diamonds). In the inset the saturation effect for the VAP tension coefficient K_0 with increasing DDI strength Λ is shown. The functional dependence $K_0 = K_0(\Lambda)$ forms the input for the evaluation of Eq.(10) for the critical temperature T_c .

we derive $T_c(\infty) \approx 1.3$. Note that the RG arguments of [11, 12] gave the much larger prediction $T_c(\infty) = 2\pi$. This in turn implies that it is rather difficult to account for all essential Feynman graphs in order to adequately describe the shielding effect in a RG calculation, which applies equally well to the case $\Lambda = 0$.

In conclusion, we have demonstrated both numerically and analytically using an analogy to plasma physics, that VAPs in the 2D dipolar XY-model dissociate at a critical temperature in a manner familiar from the BKT transition. This is due to the linear confinement potential in an isolated large VAP being shielded by a gas of small VAPs, in which the large pair becomes immersed around the transition point. Therefore, the logarithmic attraction between vortices in large pairs is restored. By obtaining a physically transparent scenario, we have therefore provided an unambiguous proof that the BKT mechanism is applicable to a much broader class of systems than hitherto established.

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Supplementary material

We studied the thermodynamics of our model using Langevin Dynamics (LD) [S1]. Using a discretized representation of the Hamiltonian at every grid point $\alpha = 1, 2, \dots, N_g$, we performed a fixed temperature run of sufficient length to get reliable averages. The calculations were performed using periodic boundary conditions on square lattices $L \times L$ with the number of independent nodes $N_g = L^2$. The LD dynamical equations are given

by:

$$\frac{ds_\alpha}{dt} = -\gamma \frac{\partial}{\partial s_\alpha} (G_0 + G_{dd}) + \zeta_\alpha,$$

where γ is a constant that determines the time scale of relaxation. The stochastic thermal noise terms satisfy $\langle \zeta_\alpha(t) \rangle = 0$ and $\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle = 2T\gamma \delta_{\alpha\beta} \delta(t - t')$. In the LD simulations we use a second-order Runge-Kutta algorithm. The equations of motion above are integrated numerically with the sufficiently small discrete time step $\Delta t = 0.005$. To compute the DDI term (2) in an efficient $\mathcal{O}(N_g \ln N_g)$ way we used a NumPy FFTW realization [S2].

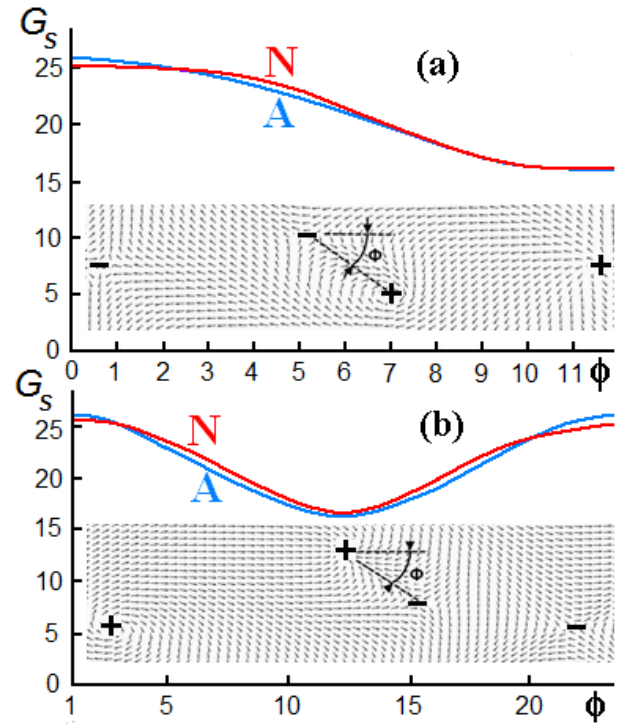


Figure 3: Free energy of a uniformly polarized 2D spin system with a small VAP in the center of a large VAP (a) and for a slightly shifted small VAP (b) versus the angle (in units $\pi/12$) between the axis of the small pair and the x axis. Red (N) and blue (A) curves correspond to the numerical calculation and the analytical approximation (3), respectively; $\Lambda = 0.2$.

The software was first used to check the assumptions leading to Eq. (3). First we consider the case of zero temperature, $T = 0$, and checked the additivity rule (3) numerically by using the imaginary time relaxation method, which is capable to find local minima of the free energy functional on a configuration space of 2D vectors s_α taken in nodes of a dense square lattice. Following [S3], the initial approximation was specified by the complex-valued skyrmion type expression $s_0(z) = 2W / (1 + |W|^2)$, with $W = \prod_{j=1}^{2N} (z - z_j)^{q_j}$, and $z = x + iy$, $z_j = x_j + iy_j$.

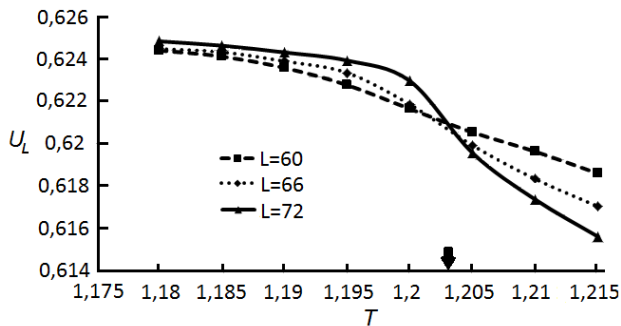


Figure 4: Explanation of Binder's approach to obtain the critical temperature [S4]. Binder's parameter U_L was calculated for $\Lambda = 0.2$; the intersection point for different lattice sizes L yields T_c (indicated by the bold arrow).

For a gas of pairs the total charge vanishes, $\sum_j q_j = 0$. Therefore, $W \rightarrow 1$ at $z \rightarrow \infty$, which agrees with the physical boundary condition $\mathbf{s} \rightarrow (1, 0)$ at $r \rightarrow \infty$. At the first stage of imaginary time propagation the cores of the VAPs begin very slowly to approach which, finally, leads to their annihilation. Due to this mutual attraction, the VAP therefore is not a real local minimum. Any initial configuration inevitably transfers at $T = 0$ to the absolute ground state $\mathbf{s} \rightarrow (1, 0)$. To stabilize VAPs, "pinning" of vortices was used by adding to the free energy a term $G_{\text{pin}} = \sum_j G_j$, $G_j = \int d\mathbf{r} V_j(\mathbf{r}) [\mathbf{s}(\mathbf{r}) - \mathbf{s}_0(\mathbf{r})]^2$, where $V_j(\mathbf{r}) = V_0 \exp[-(\mathbf{r} - \mathbf{r}_j)^2/a^2]$, $a \sim 1$. We investigated multiple configurations with different numbers of VAPs using the expression (3), and reproduced nu-

merically the total energy within a small error, less than $\sim 5\%$ (examples of typical distributions are presented in Fig. 3). The string tension constant K_0 was calculated by analyzing the forces, acting on vortices in a VAP by calculating averages of the derivatives of the pinning potential with respect to the positions of the vortices. The accuracy we were able to achieve is limited by the perturbations introduced by the pinning potential, and is sufficient to prove the reliability of the additivity rule Eq. (3), also cf. Fig. 3.

To explore the critical behavior of the model depending on the DDI coupling constant Λ numerically we use Binder's method [S4] and calculate the parameter $U_L = 1 - \langle \mathbf{s}^4 \rangle / 3 \langle \mathbf{s}^2 \rangle^2$ ("Binder's cumulant") versus temperature. Here $\mathbf{s} = \mathbf{S}/N_g$, $\mathbf{S} = \sum_{\alpha=1}^{N_g} \mathbf{s}_\alpha$ and the statistical averaging is done in a manner equivalent to the average over polarization configurations obtained with LD. The intersection point of Binder's cumulants for different values of the system size L gives T_c , as shown in Fig. 4.

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